

# Lyapunov functions for nonuniform exponential dichotomy in Banach spaces

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## Abstract

The aim of this paper is to give two integral characterizations of a concept of nonuniform exponential dichotomy, inspired by the work of L. Barreira and C. Valls. Thus, we obtain a Datko's type theorem and respectively a Lyapunov's type theorem for this concept. Some illustrative examples are given.

*Keywords:* Evolution families, Nonuniform exponential dichotomies, Lyapunov functions

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## 1. Introduction

The notion of exponential dichotomy introduced by Perron in (26) plays a central role in the qualitative theory of differential equations, dynamical systems and many other domains. The exponential dichotomy property for linear differential equations has gained prominence since the appearance of two fundamental monographs due to Daleckiĭ and Kreĭn (13) and Massera and Schäffer (20). These were followed by the important book of Coppel (12) who synthesized and improved the results that existed in the literature up to 1978. Several important papers on this subject appeared afterward and we only mention (1, 16, 17, 23, 25, 27, 29, 31–33). We also refer to the book of Chicone and Latushkin (10) for important results in infinite-dimensional

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spaces. In (30), Sacker and Sell use a concept of exponential dichotomy for skew-product semiflows with the restriction that the unstable subspace is finite dimensional. Chow and Leiva introduce in (11) a general concept of exponential dichotomy for linear skew-product semiflows weaker than the one used by Sacker and Sell. We note that all these works consider only the uniform case.

The concept of uniform exponential dichotomy has many generalizations, as for example the nonuniform case, where a consistent contribution is due to Barreira and Valls (3–8). Their study is motivated by ergodic theory and (nonuniform) hyperbolic theory (we refer the reader to (2) for details and further information). Preda and Megan introduced in (28) a concept of nonuniform exponential dichotomy which requires nothing about the norms of the associated projections. In a recent work (22), the authors involve a more general type of exponential dichotomy, but with the same assumption on the dichotomy projections.

In this paper we consider a notion of nonuniform exponential dichotomy for abstract evolution families close to that of Barreira and Valls, but we require nothing about the norms of the dichotomy projections (as in (22) and (28)). Therefore, our approaches can be extended to nonlinear evolution families. The main objective is to obtain necessary conditions and sufficient ones for the existence of this type of exponential dichotomy, our study being motivated by the recent and significant work of Barreira and Valls in the context of nonuniform behavior.

Our first main result (Theorem 1) points out a version of a well-known theorem due to Datko for uniform exponential stability (14) and to Preda and Megan for uniform exponential dichotomy (29). We note that this result was extended in the case of nonuniform exponential dichotomy in (28), but the authors use a different notion of dichotomy. Theorem 1 is used to obtain a sufficient condition for the existence of nonuniform exponential dichotomy in terms of Lyapunov function (Theorem 3). This result is close in spirit to that of Megan and Buşe (21), who consider related problems in the particular case of uniform exponential dichotomy for observable evolution operators. The importance of Lyapunov functions is well-known, particularly in the study of exponential behavior of solutions of differential equations both in finite and infinite-dimensional settings (we refer to the books (10, 12, 13, 19, 20, 24)). Other important Lyapunov's type characterizations were obtained in (9, 15, 21), but these results consider only the case of uniform exponential behavior. In the more recent papers (4–6), Barreira and Valls construct

explicitly Lyapunov functions or sequences for a given dynamics in the case of nonuniform behavior, but none of the results in (4–6) and in our paper imply the results in the other.

## 2. Exponential dichotomies

Let  $X$  be a real or complex Banach space and  $\mathcal{B}(X)$  be the Banach algebra of all bounded linear operators on  $X$ . The norms on  $X$  and on  $\mathcal{B}(X)$  will be denoted by  $\|\cdot\|$ .

**Definition 1.** A family of bounded linear operators  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  on  $X$  is called an *evolution family* if the following conditions are satisfied:

1.  $U(t, t) = Id$  (the identity operator on  $X$ ) for every  $t \geq 0$ ;
2.  $U(t, s)U(s, t_0) = U(t, t_0)$ , for all  $t \geq s \geq t_0 \geq 0$ ;
3.  $(t, s) \mapsto U(t, s)x$  is continuous for every  $x \in X$ .

**Definition 2.** A strongly continuous function  $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  is said to be a *projection valued function* if  $P^2(t) = P(t)$ , for every  $t \geq 0$ .

If  $P(\cdot)$  is a projection valued function, we denote by  $Q(t) = Id - P(t)$ , the complementary projection of  $P(t)$ . One can easily see that

$$P(t)Q(t) = Q(t)P(t) = 0, \text{ for every } t \geq 0.$$

**Definition 3.** Given an evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ , we say that a projection valued function  $P : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  is *compatible with*  $\mathcal{U}$  if the following conditions hold:

1.  $P(t)U(t, s) = U(t, s)P(s)$ , for all  $t \geq s \geq 0$ ;
2. The restriction  $U(t, s)|_{Q(s)X} : Q(s)X \rightarrow Q(t)X$  is an isomorphism for  $t \geq s \geq 0$ ; we denote its inverse by  $U_Q(s, t)$ ;
3. There exist constants  $M \geq 1$ ,  $\varepsilon \geq 0$  and  $\omega > 0$  such that

$$\|U_P(t, s)x\| \leq Me^{\varepsilon s} e^{\omega(t-s)} \|x\|, \text{ for } t \geq s \geq 0 \text{ and } x \in P(s)X$$

and

$$\|U_Q(s, t)x\| \leq Me^{\varepsilon s} e^{\omega(t-s)} \|x\|, \text{ for } t \geq s \geq 0 \text{ and } x \in Q(t)X,$$

where  $U_P(t, s)$  denotes  $U(t, s)P(s)$ , for  $t \geq s \geq 0$ .

**Definition 4.** We say that an evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  admits an *exponential dichotomy* if there exist a projection valued function  $P(\cdot)$  compatible with  $\mathcal{U}$  and constants  $N_i \geq 1$ ,  $\alpha_i \geq 0$  and  $\nu_i > 0$ ,  $i = 1, 2$  with  $\alpha_2 < \nu_2$  such that

1.  $\|U_P(t, s)x\| \leq N_1 e^{\alpha_1 s} e^{-\nu_1(t-s)} \|x\|$ , for all  $t \geq s \geq 0$  and  $x \in P(s)X$ ,
2.  $\|U_Q(s, t)x\| \leq N_2 e^{\alpha_2 t} e^{-\nu_2(t-s)} \|x\|$ , for all  $t \geq s \geq 0$  and  $x \in Q(t)X$ .

The constants  $\alpha_1$  and  $\alpha_2$  measure the non-uniformity of the dichotomy. In particular, when  $\alpha_1 = \alpha_2 = 0$  we say that  $\mathcal{U}$  has a *uniform exponential dichotomy*. The function  $P(\cdot)$  is called the *dichotomy projection*.

*Remark 1.* An evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  admits an exponential dichotomy iff there exist a projection valued function  $P(\cdot)$  compatible with  $\mathcal{U}$  and constants  $N \geq 1$ ,  $\alpha \geq 0$  and  $\nu > 0$  such that

$$\|U_P(t, s)x\| \leq N e^{\alpha s} e^{-\nu(t-s)} \|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in P(s)X$$

and

$$\|U_Q(s, t)x\| \leq N e^{\alpha s} e^{-\nu(t-s)} \|x\|, \text{ for all } t \geq s \geq 0 \text{ and } x \in Q(t)X.$$

If an evolution family  $\mathcal{U}$  has a uniform exponential dichotomy then it also has an exponential dichotomy. Even in finite-dimensional case the converse may not be true.

*Example 1.* The evolution family

$$U(t, s)(x_1, x_2) = \left( \frac{u(s)}{u(t)} x_1, \frac{u(t)}{u(s)} x_2 \right), \text{ for } t \geq s \geq 0 \text{ and } (x_1, x_2) \in \mathbb{R}^2,$$

where  $u(t) = e^{t(3+\cos^2 t)}$  for  $t \geq 0$ , has an exponential dichotomy on  $\mathbb{R}^2$  (endowed with the euclidian norm) that is not a uniform exponential dichotomy.

We first prove that  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  has exponential dichotomy with the dichotomy projection  $P(t)(x_1, x_2) = (x_1, 0)$ . It is easy to verify that  $P(\cdot)$  is compatible with  $\mathcal{U}$ . Furthermore, we have

$$\|U_P(t, s)(x, 0)\| = e^{-3(t-s)-t \cos^2 t + s \cos^2 s} |x| \leq e^s e^{-3(t-s)} \|(x, 0)\|$$

and, respectively

$$\|U_Q(s, t)(0, x)\| = e^{-4(t-s)+t \sin^2 t - s \sin^2 s} |x| \leq e^t e^{-4(t-s)} \|(0, x)\|$$

for all  $t \geq s \geq 0$  and  $x \in \mathbb{R}$ . If we now assume that  $\mathcal{U}$  has uniform exponential dichotomy with  $P(t)(x_1, x_2) = (x_1, 0)$  then there exist  $N \geq 1$  and  $\nu > 0$  such that

$$\| U_P(t, s)(x, 0) \| \leq N e^{-\nu(t-s)} |x|, \text{ for all } t \geq s \geq 0 \text{ and } x \in \mathbb{R}.$$

In particular, for  $x = 1$ ,  $t = n\pi + \pi/2$  and  $s = n\pi$  with  $n \in \mathbb{N}$ , we have

$$e^{-3\pi/2+n\pi} \leq N e^{-\nu\pi/2}, \text{ for all } n \in \mathbb{N}.$$

Letting  $n \rightarrow \infty$  in the above relation, we obtain a contradiction. Hence,  $\mathcal{U}$  has an exponential dichotomy that is not a uniform one.

We remark that the notion of exponential dichotomy from above requires nothing about the norms of the projections. More precisely, even in finite-dimensional spaces the norms of the projections can be made arbitrarily large and, in contrast with the uniform case, we can obtain that  $\| P(t) \| \rightarrow \infty$  as  $t \rightarrow \infty$ . We give an explicit example:

*Example 2.* Let  $a > 0$  be a real parameter and consider

$$V(t, s) = S(t)U(t, s)S(s)^{-1}, \quad t \geq s \geq 0,$$

where  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  is the evolution family in Example 1 and

$$S(t)(x_1, x_2) = \left( x_1 + \frac{t+a}{\sqrt{1+(t+a)^2}} x_2, \frac{1}{\sqrt{1+(t+a)^2}} x_2 \right).$$

For each  $t \geq 0$  and  $(x_1, x_2) \in \mathbb{R}^2$ , we take  $\tilde{P}(t)(x_1, x_2) = (x_1 - (t+a)x_2, 0)$ .

Clearly,  $\tilde{P}(\cdot)$  is a projection valued function compatible with the evolution family  $\{V(t, s)\}_{t \geq s \geq 0}$ .

A simple computation shows that  $\| S(t) \| \leq \sqrt{2}$ ,  $S(t)^{-1}P(t) = P(t)$  and  $P(t)\tilde{P}(t) = \tilde{P}(t)$ , for  $t \geq 0$ , where  $P(t)(x_1, x_2) = (x_1, 0)$ .

Since  $\mathcal{U}$  has exponential dichotomy with the dichotomy projection  $P(\cdot)$ ,

it follows that for all  $t \geq s \geq 0$  and  $x \in \text{Range} \tilde{P}(s)$  we have

$$\begin{aligned} \| V_{\tilde{P}}(t, s)x \| &= \| S(t)U(t, s)S(s)^{-1}P(s)\tilde{P}(s)x \| \\ &\leq \sqrt{2} \| U_P(t, s)P(s)\tilde{P}(s)x \| \\ &\leq \sqrt{2}e^s e^{-3(t-s)} \| x \| . \end{aligned}$$

On the other hand, since the complementary projection of  $\tilde{P}(t)$  is given by  $\tilde{Q}(t)(x_1, x_2) = ((t+a)x_2, x_2)$  and

$$\text{Range } \tilde{Q}(t) = \left\{ \left( \frac{\alpha(t+a)}{\sqrt{1+(t+a)^2}}, \frac{\alpha}{\sqrt{1+(t+a)^2}} \right) : \alpha \in \mathbb{R} \right\},$$

we deduce that

$$\begin{aligned} \| V_{\tilde{Q}}(s, t)x \| &= \left\| S(s)U(s, t)S(t)^{-1} \left( \frac{\alpha(t+a)}{\sqrt{1+(t+a)^2}}, \frac{\alpha}{\sqrt{1+(t+a)^2}} \right) \right\| \\ &= \| S(s)U_Q(s, t)(0, \alpha) \| \leq \sqrt{2}e^t e^{-4(t-s)} |\alpha| \\ &= \sqrt{2}e^t e^{-4(t-s)} \| x \|, \end{aligned}$$

for all  $t \geq s \geq 0$  and  $x \in \text{Range } \tilde{Q}(t)$ . Hence,  $\{V(t, s)\}_{t \geq s \geq 0}$  has exponential dichotomy with the dichotomy projection  $\tilde{P}(\cdot)$ . On the other hand, we have

$$\| \tilde{P}(t) \| = \sup_{\|(x_1, x_2)\|=1} |x_1 - (t+a)x_2| \geq t+a. \quad (1)$$

By (1), we deduce that  $\|\tilde{P}(t)\| \rightarrow \infty$  as  $t \rightarrow \infty$  and for each  $t \geq 0$  the norm of the projection  $\tilde{P}(t)$  can be made arbitrarily large by making  $a \rightarrow \infty$ .

*Remark 2.* An evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  admits an exponential dichotomy iff there exist a projection valued function  $P(\cdot)$  compatible with  $\mathcal{U}$  and constants  $N \geq 1$ ,  $\alpha \geq 0$  and  $\nu > 0$  such that

$$\| U(t, t_0)P(t_0)x \| \leq Ne^{\alpha s} e^{-\nu(t-s)} \| U(s, t_0)P(t_0)x \|$$

and

$$\| U(s, t_0)Q(t_0)x \| \leq Ne^{\alpha s} e^{-\nu(t-s)} \| U(t, t_0)Q(t_0)x \|$$

for all  $t \geq s \geq t_0 \geq 0$  and  $x \in X$ .

Setting  $s = t_0$  in the above relations, we obtain that if  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  has exponential dichotomy with the dichotomy projection  $P(\cdot)$ , then for any  $t_0 \geq 0$  and  $x_0 \in X$ , we have

$$\|U(t, t_0)P(t_0)x_0\| \rightarrow 0 \text{ as } t \rightarrow \infty, \quad (2)$$

and

$$\|U(t, t_0)Q(t_0)x_0\| \rightarrow \infty \text{ as } t \rightarrow \infty, \text{ unless } Q(t_0)x_0 = 0. \quad (3)$$

Therefore, we can say that the concept of nonuniform exponential dichotomy considered in this paper is a natural extension of the uniform one, although we require nothing about the norms of the projections. We point out that for the nonuniform exponential dichotomies studied in (3–8) it follows that the norms of the dichotomy projections cannot increase faster than an exponential, but relation (3) from above is not necessarily valid.

*Example 3.* Consider the evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  in  $\mathbb{R}^2$  given by

$$U(t, s)x = e^{-(t-s)}x, \text{ for } t \geq s \geq 0 \text{ and } x \in \mathbb{R}^2.$$

$\mathcal{U}$  admits a nonuniform exponential dichotomy in the sense of Definition 2.1 in (3, pp. 20), and  $U(t, t_0)x_0 \rightarrow 0$  as  $t \rightarrow \infty$ , for all  $t_0 \geq 0$  and  $x_0 \in \mathbb{R}^2$ .

Indeed, if we set the projections  $P(t)(x_1, x_2) = (x_1, 0)$ , it follows that

$$\|U(t, s)P(s)\| \leq e^{-(t-s)} \text{ and } \|U(t, s)^{-1}Q(t)\| \leq e^{-(t-s)+2t},$$

for all  $t \geq s \geq 0$ , where  $Q(t)(x_1, x_2) = (0, x_2)$ . Thus, (2.4) and (2.5) in (3, pp. 20) hold. On the other hand, the evolution family  $\mathcal{U}$  does not admit exponential dichotomy with  $P(t)(x_1, x_2) = (x_1, 0)$ .

*Remark 3.* Examples 2 and 3 show that the exponential dichotomy in our paper and the one considered by Barreira and Valls in (3–8) are different concepts.

### 3. The main results

Given an evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  and a projection valued function  $P(\cdot)$  compatible with  $\mathcal{U}$ , for simplicity we assume that  $U_Q(s, t)$  is defined on the whole space  $X$ , which can be done by identifying this operator with  $U_Q(s, t)Q(t)$ , for  $t \geq s \geq 0$ .

The next result gives a necessary condition for the existence of exponential dichotomy of an evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$ :

**Proposition 1.** *Let  $p > 0$  be a real constant. If the evolution family  $\mathcal{U}$  has exponential dichotomy with the dichotomy projection  $P(\cdot)$  then there exist three constants  $K \geq 1$ ,  $\gamma > 0$  and  $\beta \geq 0$  such that*

$$\begin{aligned} \int_t^\infty e^{p\gamma(\tau-t)} \|U_P(\tau, t)x\|^p d\tau + \int_0^t e^{p\gamma(t-\tau)} \|U_Q(\tau, t)x\|^p d\tau \\ \leq K e^{p\beta t} (\|P(t)x\|^p + \|Q(t)x\|^p) \end{aligned} \quad (4)$$

for all  $t \geq 0$  and  $x \in X$ .

*Proof.* It is a simple exercise, setting  $\beta = \max\{\alpha_1, \alpha_2\}$ ,  $\gamma \in (0, \min\{\nu_1, \nu_2\})$  and  $K = \max\left\{\frac{N_1^p + N_2^p}{p(\nu - \gamma)}, 1\right\}$ , where  $N_1, N_2 \geq 1$ ,  $\nu_1, \nu_2 > 0$  and  $\alpha_1, \alpha_2 \geq 0$  are given by Definition 4, and  $\nu = \min\{\nu_1, \nu_2\}$ .  $\square$

The example below shows that the converse of the above result may not be valid:

*Example 4.* Consider the evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  in  $\mathbb{R}^2$  given by

$$U(t, s)(x_1, x_2) = \left( \frac{u_1(s)}{u_1(t)} x_1, \frac{u_2(t)}{u_2(s)} x_2 \right),$$

where  $u_1(t) = e^{t(1+\cos^2 t)}$  and  $u_2(t) = e^{t \cos^2 t}$ , and also consider the projection valued function  $P : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}^2)$  defined by  $P(t)(x_1, x_2) = (x_1, 0)$ . Clearly,  $Q(t)(x_1, x_2) = (0, x_2)$  and the restriction  $U(t, s)|_{Q(s)\mathbb{R}^2} : Q(s)\mathbb{R}^2 \rightarrow Q(t)\mathbb{R}^2$  is an isomorphism for  $t \geq s \geq 0$ . Moreover, we have that

$$\|U_P(t, s)(x, 0)\| \leq e^s e^{t-s}|x| \text{ and } \|U_Q(s, t)(0, x)\| \leq e^s e^{t-s}|x|$$

for  $t \geq s \geq 0$  and  $x \in \mathbb{R}$ . Thus, the projection valued function  $P(\cdot)$  is compatible with  $\mathcal{U}$ .



For any real parameter  $p > 0$ , we have

$$\begin{aligned}
& \int_t^\infty e^{\frac{1}{2}p(\tau-t)} \|U_P(\tau, t)x\|^p d\tau + \int_0^t e^{\frac{1}{2}p(t-\tau)} \|U_Q(\tau, t)x\|^p d\tau = \\
& = \int_t^\infty e^{-\frac{1}{2}p(\tau-t)} e^{-p\tau \cos^2 \tau + pt \cos^2 t} d\tau |x_1|^p + \int_0^t e^{-\frac{1}{2}p(t-\tau)} e^{-p\tau \sin^2 \tau + pt \sin^2 t} d\tau |x_2|^p \\
& \leq e^{pt} \int_t^\infty e^{-\frac{1}{2}p(\tau-t)} d\tau |x_1|^p + e^{pt} \int_0^t e^{-\frac{1}{2}p(t-\tau)} d\tau |x_2|^p \\
& \leq (2/p)e^{pt} (\|P(t)x\|^p + \|Q(t)x\|^p), \text{ for } t \geq 0 \text{ and } x = (x_1, x_2) \in \mathbb{R}^2.
\end{aligned}$$

Hence, (4) holds. If we assume that the evolution family  $\mathcal{U}$  has exponential dichotomy with the dichotomy projection  $P(\cdot)$ , then there exist  $N \geq 1$ ,  $\alpha \geq 0$  and  $\nu > 0$  such that

$$\|U_Q(s, t)x\| \leq Ne^{\alpha s} e^{-\nu(t-s)} \|x\|,$$

for all  $t \geq s \geq 0$  and  $x = (0, x_2) \in \mathbb{R}^2$ . Letting  $t = 2n\pi + \frac{\pi}{2}$  with  $n \in \mathbb{N}$ ,  $s = 0$  and  $x = (0, 1)$ , we obtain  $e^{(2n\pi + \pi/2)\nu} \leq N$ , for all  $n \in \mathbb{N}$ , which is false. Thus, the evolution family  $\mathcal{U}$  does not admit exponential dichotomy with  $P(t)(x_1, x_2) = (x_1, 0)$ .

The question is: What additional properties must the constants  $\gamma$  and  $\beta$  possess such that (4) implies the existence of exponential dichotomy for the evolution family  $\mathcal{U}$ ?

To answer this question we now come to our first main result, which is a Datko's type theorem for nonuniform exponential dichotomy. Variants of the theorem below are given in (18) for weak exponential stability of evolution families and in (34) for uniform exponential stability of skew-evolution semiflows, but using different approaches.

**Theorem 1.** *Let  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  be an evolution family and  $P(\cdot)$  be a projection valued function compatible with  $\mathcal{U}$ . If there exist  $p > 0$ ,  $K \geq 1$ ,  $\gamma > \varepsilon$  and  $\beta \in [0, \gamma)$  such that (4) holds then  $\mathcal{U}$  has an exponential dichotomy.*

*Proof.* Let  $t \geq s \geq 0$ . First, we consider  $x \in P(s)X$ . If  $t \geq s + 1$  then we

have

$$\begin{aligned}
e^{p\gamma(t-s)} \| U_P(t, s)x \|^p &= \int_{t-1}^t e^{p\gamma(t-s)} \| U_P(t, s)x \|^p d\tau \\
&\leq M^p e^{p\varepsilon t} \int_{t-1}^t e^{p(\gamma+\omega)(t-\tau)} e^{p\gamma(\tau-s)} \| U_P(\tau, s)x \|^p d\tau \\
&\leq M^p e^{p(\gamma+\omega)} e^{p\varepsilon t} \int_s^\infty e^{p\gamma(\tau-s)} \| U_P(\tau, s)x \|^p d\tau \\
&\leq KM^p e^{p(\gamma+\omega)} e^{p\beta s} e^{p\varepsilon t} \| x \|^p.
\end{aligned}$$

Therefore,

$$\| U_P(t, s)x \| \leq K^{1/p} M e^{\gamma+\omega} e^{(\beta+\varepsilon)s} e^{-(\gamma-\varepsilon)(t-s)} \| x \|, \text{ for } t \geq s+1. \quad (5)$$

For  $t \in [s, s+1)$ , it follows that

$$\| U_P(t, s)x \| \leq M e^{\varepsilon s} e^{\omega(t-s)} \| x \| \leq M e^{\omega} e^{\varepsilon s} \| x \|. \quad (6)$$

By (5) and (6), we obtain that there is  $N_1 \geq 1$  such that

$$\| U_P(t, s)x \| \leq N_1 e^{(\beta+\varepsilon)s} e^{-(\gamma-\varepsilon)(t-s)} \| x \|, \quad (7)$$

for all  $t \geq s \geq 0$  and  $x \in P(s)X$ .

Let now  $x \in Q(t)X$ . We have

$$\begin{aligned}
e^{p\gamma(t-s)} \| U_Q(s, t)x \|^p &= \int_s^{s+1} e^{p\gamma(t-s)} \| U_Q(s, t)x \|^p d\tau \\
&\leq M^p e^{p\varepsilon s} \int_s^{s+1} e^{p(\gamma+\omega)(\tau-s)} e^{p\gamma(t-\tau)} \| U_Q(\tau, t)x \|^p d\tau \\
&\leq M^p e^{p(\gamma+\omega)} e^{p\varepsilon s} \int_0^t e^{p\gamma(t-\tau)} \| U_Q(\tau, t)x \|^p d\tau \\
&\leq KM^p e^{p(\gamma+\omega)} e^{p\varepsilon s} e^{p\beta t} \| x \|^p, \text{ for } t \geq s+1.
\end{aligned}$$

This shows that

$$\| U_Q(s, t)x \| \leq K^{1/p} M e^{\gamma+\omega} e^{(\beta+\varepsilon)t} e^{-(\gamma+\varepsilon)(t-s)} \| x \|, \text{ for } t \geq s+1.$$

Moreover, we can easily verify that there is  $N_2 \geq 1$  such that

$$\| U_Q(s, t)x \| \leq N_2 e^{(\beta+\varepsilon)t} e^{-(\gamma+\varepsilon)(t-s)} \| x \|, \quad (8)$$

for all  $t \geq s \geq 0$  and  $x \in Q(t)X$ . Finally, (7) and (8) involve that  $\mathcal{U}$  has an exponential dichotomy. This ends the proof.  $\square$

*Remark 4.* Example 4 shows that the conditions  $\gamma > \varepsilon$  and  $\beta \in [0, \gamma)$  in the above theorem are essential.

Given a real constant  $\gamma > 0$  and a projection valued function  $P(\cdot)$ , we denote by  $\mathcal{H}_\gamma(P)$  the set of all strongly continuous operator-valued function  $H : \mathbb{R}_+ \longrightarrow \mathcal{B}(X)$  with

$$\| H(t)x \| \leq e^{\gamma t} \| P(t)x \| + e^{-\gamma t} \| Q(t)x \|, \text{ for all } (t, x) \in \mathbb{R}_+ \times X. \quad (9)$$

Let  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  be an evolution family,  $P(\cdot)$  be a projection valued function compatible with  $\mathcal{U}$  and let  $H \in \mathcal{H}_\gamma(P)$ . We say that a continuous function  $L : \mathbb{R}_+ \times X \longrightarrow \mathbb{R}$  is a *Lyapunov function* corresponding to  $\mathcal{U}$ ,  $P(\cdot)$  and  $H(\cdot)$  if

$$L(t, U(t, s)x) + \int_s^t \| H(\tau)U(\tau, s)x \|^2 d\tau \leq L(s, x), \quad (10)$$

for all  $t \geq s \geq 0$  and  $x \in X$ .

The following result shows that the existence of an exponential dichotomy implies the existence of a Lyapunov function.

**Theorem 2.** *If an evolution family  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  admits exponential dichotomy with the dichotomy projection  $P(\cdot)$  then there exist  $K \geq 1$ ,  $\gamma > 0$  and  $\beta \geq 0$  such that for each  $H \in \mathcal{H}_\gamma(P)$  there is a Lyapunov function  $L$  which satisfies the following properties:*

- (L<sub>1</sub>)  $|L(t, x)| \leq K (e^{2(\gamma+\beta)t} \| P(t)x \|^2 + e^{-2(\gamma-\beta)t} \| Q(t)x \|^2)$
- (L<sub>2</sub>)  $L(t, P(t)x) \geq 0$  and  $L(t, Q(t)x) \leq 0$ , for all  $(t, x) \in \mathbb{R}_+ \times X$ .

*Proof.* Setting  $p = 2$  in Proposition 1, there exist  $K \geq 1$ ,  $\gamma > 0$  and  $\beta \geq 0$  such that (4) holds. For each  $H \in \mathcal{H}_\gamma(P)$ , taking

$$L(t, x) = 2 \int_t^\infty \| H(\tau)U_P(\tau, t)x \|^2 d\tau - 2 \int_0^t \| H(\tau)U_Q(\tau, t)x \|^2 d\tau,$$

we obtain

$$\begin{aligned}
L(t, U(t, s)x) &+ \int_s^t \| H(\tau)U(\tau, s)x \|^2 d\tau \\
&= 2 \int_t^\infty \| H(\tau)U_P(\tau, s)x \|^2 d\tau - 2 \int_0^t \| H(\tau)U_Q(\tau, s)x \|^2 d\tau \\
&\quad + \int_s^t \| H(\tau)U_P(\tau, s)x + H(\tau)U_Q(\tau, s)x \|^2 d\tau \\
&\leq 2 \int_s^\infty \| H(\tau)U_P(\tau, s)x \|^2 d\tau - 2 \int_0^s \| H(\tau)U_Q(\tau, s)x \|^2 d\tau \\
&= L(s, x),
\end{aligned}$$

for all  $t \geq s \geq 0$  and  $x \in X$ . Therefore,  $L$  is a Lyapunov function. Moreover, for all  $(t, x) \in \mathbb{R}_+ \times X$ , we have

$$L(t, P(t)x) = 2 \int_t^\infty \| H(\tau)U_P(\tau, t)x \|^2 d\tau \geq 0,$$

and, respectively

$$L(t, Q(t)x) = -2 \int_0^t \| H(\tau)U_Q(\tau, t)x \|^2 d\tau \leq 0.$$

By (9) and Proposition 1, we deduce that

$$L(t, P(t)x) \leq 2 \int_t^\infty e^{2\gamma\tau} \| U_P(\tau, t)x \|^2 d\tau \leq 2Ke^{2(\gamma+\beta)t} \| P(t)x \|^2,$$

and

$$|L(t, Q(t)x)| \leq 2 \int_0^t e^{-2\gamma\tau} \| U_Q(\tau, t)x \|^2 d\tau \leq 2Ke^{-2(\gamma-\beta)t} \| Q(t)x \|^2,$$

for all  $(t, x) \in \mathbb{R}_+ \times X$ . This completes the proof.  $\square$

**Theorem 3.** *Let  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  be an evolution family and  $P(\cdot)$  be a projection valued function compatible with  $\mathcal{U}$ . If there exist  $K \geq 1$ ,  $\gamma > \varepsilon$  and  $\beta \in [0, \gamma)$  such that for each  $H \in \mathcal{H}_\gamma(P)$  there is a Lyapunov function  $L$  which satisfies  $(L_1)$  and  $(L_2)$ , then  $\mathcal{U}$  has an exponential dichotomy.*

*Proof.* For  $\gamma > \varepsilon$ , letting

$$H(t)x = e^{\gamma t}P(t)x + e^{-\gamma t}Q(t)x, \text{ for } (t, x) \in \mathbb{R}_+ \times X,$$

we have

$$\begin{aligned} & \int_t^u e^{2\gamma(\tau-t)} \|U_P(\tau, t)x\|^2 d\tau + \int_0^t e^{2\gamma(t-\tau)} \|U_Q(\tau, t)x\|^2 d\tau \\ &= e^{-2\gamma t} \int_t^u \|H(\tau)U_P(\tau, t)x\|^2 d\tau + e^{2\gamma t} \int_0^t \|H(\tau)U_Q(\tau, t)x\|^2 d\tau \\ &\leq e^{-2\gamma t} (L(t, P(t)x) - L(u, U_P(u, t)x)) + e^{2\gamma t} (L(0, U_Q(0, t)x) - L(t, Q(t)x)) \\ &\leq e^{-2\gamma t} |L(t, P(t)x)| + e^{2\gamma t} |L(t, Q(t)x)| \leq Ke^{2\beta t} (\|P(t)x\|^2 + \|Q(t)x\|^2) \end{aligned}$$

for all  $u \geq t \geq 0$  and  $x \in X$ .

Therefore,

$$\begin{aligned} & \int_t^\infty e^{2\gamma(\tau-t)} \|U_P(\tau, t)x\|^2 d\tau + \int_0^t e^{2\gamma(t-\tau)} \|U_Q(\tau, t)x\|^2 d\tau \\ & \leq Ke^{2\beta t} (\|P(t)x\|^2 + \|Q(t)x\|^2) \end{aligned}$$

for all  $(t, x) \in \mathbb{R}_+ \times X$ . By Theorem 1, we deduce that  $\mathcal{U}$  has an exponential dichotomy.  $\square$

*Example 5.* Consider the evolution family  $\mathcal{U}$  and the projection valued function  $P(\cdot)$  in Example 1, and take  $K = 1$ ,  $\gamma = 2$  and  $\beta = 1$  ( $\gamma > \varepsilon = 1$  and  $\beta \in [0, \gamma)$ ). For each strongly continuous operator-valued function  $H : \mathbb{R}_+ \rightarrow \mathcal{B}(\mathbb{R}^2)$  with

$$\|H(t)(x_1, x_2)\| \leq e^{2t}|x_1| + e^{-2t}|x_2|, \text{ for } (t, x_1, x_2) \in \mathbb{R}_+ \times \mathbb{R}^2$$

(that is  $H \in \mathcal{H}_2(P)$ ), we set

$$L(t, x) = 2 \int_t^\infty \|H(\tau)(e^{f(t)-f(\tau)}x_1, 0)\|^2 d\tau - 2 \int_0^t \|H(\tau)(0, e^{f(\tau)-f(t)}x_2)\|^2 d\tau$$

for  $x = (x_1, x_2) \in \mathbb{R}^2$  and  $t \geq 0$ , where  $f(t) = t(3 + \cos^2 t)$ . It follows from the proof of Theorem 2 that  $L$  is a Lyapunov function. Moreover, conditions  $(L_1)$  and  $(L_2)$  hold. Thus, by Theorem 3, we deduce that  $\mathcal{U}$  has an exponential dichotomy.

When  $X$  is a Hilbert space, we get the following result:

**Corollary 1.** *Let  $\mathcal{U} = \{U(t, s)\}_{t \geq s \geq 0}$  be an evolution family on the Hilbert space  $X$  and  $P(\cdot)$  be a projection valued function compatible with  $\mathcal{U}$ . If there exist  $K \geq 1$ ,  $\gamma > \varepsilon$  and  $\beta \in [0, \gamma)$  such that for each  $H \in \mathcal{H}_\gamma(P)$  there is a strongly continuous operator-valued function  $W : \mathbb{R}_+ \rightarrow \mathcal{B}(X)$  with  $W(t)^* = W(t)$ ,  $\forall t \geq 0$  and*

1.  $\langle U(t, s)^* W(t) U(t, s)x + \int_s^t U(\tau, s)^* H(\tau)^* H(\tau) U(\tau, s)x d\tau, x \rangle \leq \langle W(s)x, x \rangle$
2.  $|\langle W(t)x, x \rangle| \leq K (e^{2(\gamma+\beta)t} \|P(t)x\|^2 + e^{-2(\gamma-\beta)t} \|Q(t)x\|^2)$
3.  $\langle W(t)P(t)x, P(t)x \rangle \geq 0$
4.  $\langle W(t)Q(t)x, Q(t)x \rangle \leq 0$

for all  $t \geq s \geq 0$  and  $x \in X$ , then  $\mathcal{U}$  has an exponential dichotomy.

*Proof.* It follows from Theorem 3, letting

$$L(t, x) = \langle W(t)x, x \rangle, \text{ for every } (t, x) \in \mathbb{R}_+ \times X.$$

□

#### 4. Open Problem

An important property of exponential dichotomies is their roughness, that is, the preservation of exponential dichotomy under perturbations of the evolution family (we refer the reader to (3, 6, 25, 27) and the references therein). We address the question of roughness for the concept of exponential dichotomy considered in this paper.

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